Some Discrete Competition Models and the Competitive Exclusion Principle†

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the system

$$_{+1} = {}_1 \frac{1}{1 + \; + {}_1} \tag{1}$$

$$_{+1} = {}_2 \frac{1}{1 + {}_2 \; + } \tag{2}$$

where > 0 and > 0. We denote solutions of this system by $(,)$, $= 0, 1, 2, 3, \ldots$. (For some results concerning difference equations defined by rational functions see Refs. [3,4,25]. The results in these papers do not apply to the Leslie/Gower model, however. Other papers that deal with discrete competition models include [16–19].)

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s ≥ 0, ≥ 0. $\mathbb{R}^2_+ \doteq [0, +\infty) \times [0, +\infty)$, $\mathring{\mathbb{R}}^2_+ \doteq (0, +\infty) \times (0, +\infty)$

$$: (,) \to \left({}_1 \frac{1}{1 + \; + {}_1}, \quad {}_2 \frac{1}{1 + {}_2 \; + } \right)$$

s $\mathbb{R}^2_+$. Th s s $\mathring{\mathbb{R}}^2_+$ s $[0, +\infty) \times 0$ 0 $\times [0, +\infty)$. 6 , ll s s $\mathbb{R}^2_+$. ll , $: \mathbb{R}^2_+ \to S \doteq [0, {}_1) \times [0, {}_2)$. ll s s 1 . [30 ll s $\mathbb{R}^2_+$ s $\to +\infty$.

Th is also invertible on $\mathbb{R}^2_+$, since for $(', ') \in S$ in the range of the equations

$$_1 \frac{1}{1 + \; + {}_1} = ', \quad {}_2 \frac{1}{1 + {}_2 \; + } = '$$

have the unique solution

$$= \frac{{}_2 - 1 + {}_1}{\Delta}, \quad = \frac{{}_1 - 1 + {}_2}{\Delta}$$

where

$$_1 \doteq \frac{1}{} > 1, \quad {}_2 \doteq \frac{2}{} > 1, \quad \Delta \doteq ({}_1 - 1)({}_2 - 1) - {}_1 {}_2$$

(the range of is defined by the inequality $\Delta > 0$). The formulas for the pre-images and show the inverse $^{-1}$ continuous.

1 $: \mathbb{R}^2_+ \to S$.

Th s $E_0 : (0, 0)$; $E_1 : ({}_1 - 1, 0)$; $E_2 : (0, {}_2 - 1)$ s (i.e. are equilibria of the Leslie/Gower model (1) and (2)). These are “exclusion” equilibria.

Th s s s -coordinate is held fixed by the map is the line $+ {}_1 = {}_1 - 1$. If this line intersects $\mathring{\mathbb{R}}^2_+$ (i.e. if $_1 > 1$), we denote the resulting line segment by $_1$. Similarly, if $_2 > 1$, the points on the line segment $_2$ from the line $_2 + = {}_2 - 1$ lying in $\mathbb{R}^2_+$ is the set of points in $\mathbb{R}^2_+$ whose -coordinate is held fixed by the map . If $_1 > 1$ the map takes a point $(,) \in \mathbb{R}^2_+$ lying above (below) $_1$ to a point with smaller (larger) -coordinate. If $_2 > 1$ the map takes a point $(,) \in \mathbb{R}^2_+$ lying above (below) $_2$ to a point with smaller (larger) -coordinate.

Th is

$$E_3 : \left(\frac{{}_2 - 1}{{}_1 {}_2 - 1} \left({}_1 - \frac{{}_1 - 1}{{}_2 - 1} \right), \quad \frac{{}_1 - 1}{{}_1 {}_2 - 1} \left({}_2 - \frac{{}_2 - 1}{{}_1 - 1} \right) \right)$$

Th[illegible] s [illegible] $\mathbb{R}^2_+$ [illegible] $_1 > 1$, $_2 > 1$ [illegible] $_1$ and $_2$ intersect in $\mathbb{R}^2_+$.
This is a “coexistence” equilibrium.

A [illegible] s [illegible] $\mathbb{R}^2_+$ (or $\mathbb{R}$ [illegible]

Cases B and C:

$$\text{Case B}: \quad {}_1({}_2 - 1) < {}_1 - 1, \quad {}_2({}_1 - 1) < {}_2 - 1$$
$$\text{Case C}: \quad {}_1({}_2 - 1) > {}_1 - 1, \quad {}_2({}_1 - 1) > {}_2 - 1. \qquad (5)$$

F... h s ... l ... s ... b

$${}_1 {}_2 < 1 \ \text{... Case B}$$
$${}_1 {}_2 > 1 \ \text{... Case C.} \qquad (6)$$

A... ${}_1 > 1$ ... ${}_2 > 1$. ... r ... E

[illegible] E_3, at which the Jacobian is

$$J_{3} = \begin{pmatrix} \dfrac{{}_{1}\,{}_{2}\,{}_{1} - {}_{1}\,{}_{2} + {}_{1} - 1}{{}_{1}({}_{1}\,{}_{2} - 1)} & {}_{1}\dfrac{{}_{1} - {}_{1}\,{}_{2} + {}_{1} - 1}{{}_{1}({}_{1}\,{}_{2} - 1)} \\ {}_{2}\dfrac{- {}_{2}\,{}_{1} + {}_{2} + {}_{2} - 1}{{}_{2}({}_{1}\,{}_{2} - 1)} & \dfrac{- {}_{2}\,{}_{1} + {}_{2}\,{}_{1}\,{}_{2} + {}_{2} - 1}{{}_{2}({}_{1}\,{}_{2} - 1)} \end{pmatrix}$$

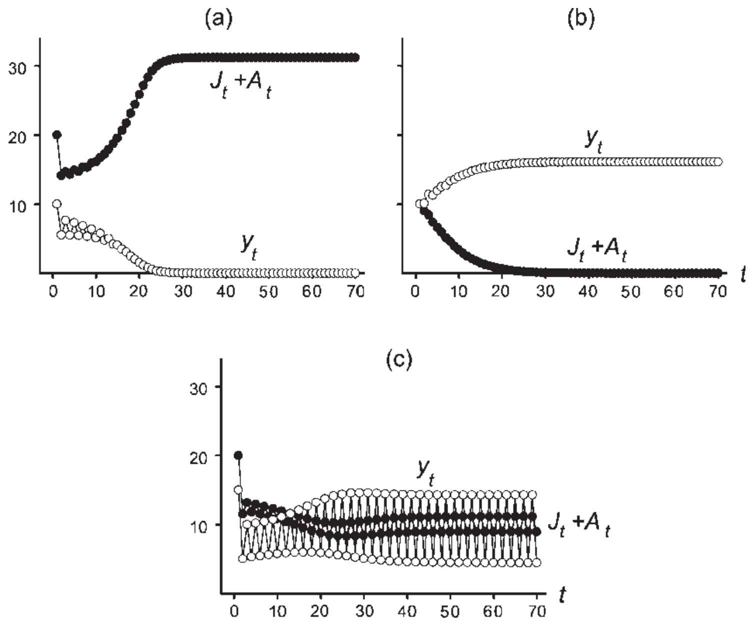
[redacted]

s s s $(1) = 1 - \mathrm{tr}\,{}_3 + \det{}_3 < 0$ (s h ( ) s 6 s C s C),
$(-1) = 1 + \mathrm{tr}\,{}_3 + \det{}_3 > 0$ (s ( ) h s), $(+\infty) = +\infty$. T h s ,

A JUVENILE/ADULT RICKER MODEL

Th ... s ... E . (7) ...

$$E_0 : (\;, A, \;) = (0, 0, 0)$$



a coexistence attractor. Figure 2 shows an example. In that figure one initial condition approaches a coexistence two-cycle, while other initial conditions lead to the extinction equilibria E_1 and E_2.

s ss 6 b s s s b s 6 F 2.
T s s b s b s (7)

$$_1(1 - \;)\; \exp(- \;_{11}(1 - \;)\; - \;_2\;_{12}\; e^{-\;_{21}\;-\;_{22}}) = \;$$

$$_1(1 - \;)A e^{-\;_{11}A-\;_{12}} = A \quad (8)$$

$$^2_2 e^{-\;_{21}\;-\;_{22}} \exp(- \;_1\;_{21}A e^{-\;_{11}A-\;_{12}} - \;_2\;_{22}\; e^{-\;_{21}\;-\;_{22}}) = \;.$$

A b b s s F . 2, b s b s
s b s, b s b b

$$(0, A, \;_1), \quad (\;, 0, \;_2) \quad (9)$$

b $> 0, A > 0,$ $> 0.$ b s s b b 6 s
6 b b s T b s (8) b $= 0$
b s

$$_1(1 - \;)e^{-\;_{11}A-\;_{12}} = 1$$

$$^2_2 e^{-\;_{21}\;-\;_{22}} \exp(- \;_1\;_{21}A e^{-\;_{11}A-\;_{12}} - \;_2\;_{22}\; e^{-\;_{21}\;-\;_{22}}) = 1$$

A and . Using the first equation in the second we can simplify the second equation

$$ {}_1(1 - \,)\, e^{-\ {}_{11}A - \ {}_{12}} = 1 $$

$$ \frac{2}{2} \left(- \ {}_{22} - \ {}_{21} \frac{1}{1 - } A - \ {}_2 \ {}_{22}\, e^{-\ {}_{22}} \right) = 1. $$

$$ A = \frac{1}{{}_{11}} (\ln \ - \ {}_{12}) \tag{10} $$

$$ {}_2 \ {}_{22}\, e^{-\ {}_{22}} = \left(\frac{{}_{12}\ {}_{21}}{(1 - \,)\ {}_{11}} - \ {}_{22} \right) + \left(2 \ln \ {}_2 - \frac{{}_{21}}{(1 - \,)\ {}_{11}} \ln \ {}_1(1 - \,) \right) \tag{11} $$

. A solution $= {}_1 > 0$, together with Eq. (10), yields the first point in a two-cycle (9). (If $A > 0$ then this point is not an equilibrium.) The Eq. (11) can be analyzed geometrically by investigating the graphs of both sides of the equation for intersection points > 0. The left hand side is a positive, one humped graph passing through $= 0$ and having $= 0$ as an asymptote. The right hand side is a straight line whose slope is positive under the assumptions in Theorem 5(b) and (c). If the -intercept of the straight line is positive, then either there are two intersection points of these graphs, no intersection point at all, or a

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